ALGEBRAIC CURVES
SOLUTIONS SHEET 8

Unless otherwise specified, k is an algebraically closed field.

Exercise 1. Let $r \geq 1$, $P \in \mathbb{A}^r_k$. Call $\mathcal{O} := \mathcal{O}_P(\mathbb{A}^r_k)$ and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$.

(1) Compute $\chi(n) = dim_k(\mathcal{O}/\mathfrak{m}^n)$ for $r = 1, 2$.
(2) For arbitrary r, show that $\chi(n)$ is a polynomial of degree r in n with leading coefficient $1/r!$.

Solution 1. Let us first make a general observation, which will be helpful for points (1) and (2). First of all, as all local rings of $\mathbb{A}^r$ are isomorphic (using translations, see Exercise 1 on Sheet 4), we may assume that $P = 0$. Denote $R = k[x_1, \ldots, x_r]$ and $\mathfrak{n} = (x_1, \ldots, x_r)$. We want to simplify $\mathcal{O}/\mathfrak{m}^n$ by using point (3) of Exercise 2 on Sheet 1; for $S = R \setminus \mathfrak{n}$ we have

$$\mathcal{O}\big/\mathfrak{m}^n = S^{-1}R\Big/S^{-1}\mathfrak{n}^n \cong (S/\mathfrak{n}^n)^{-1}\left(R\big/\mathfrak{n}^n\right)$$

Now note that $R/\mathfrak{n}^n$ is a local ring with maximal ideal $\overline{\mathfrak{n}} := \mathfrak{n}/\mathfrak{n}^n$: indeed, the prime ideals of $R/\mathfrak{n}^n$ correspond to prime ideals of R containing $\mathfrak{n}^n$, and $\mathfrak{n}$ is the only prime ideal containing $\mathfrak{n}^n$. Note also that $S/\mathfrak{n}^n \subseteq (R/\mathfrak{n}^n) \setminus \overline{\mathfrak{n}}$. In particular, all elements of $S/\mathfrak{n}^n$ (recall that this denotes the image of S under the quotient map) are already units in $R/\mathfrak{n}^n$. As localizing at a set of units has no effect, we hence obtain

$$\mathcal{O}\big/\mathfrak{m}^n \cong R\big/\mathfrak{n}^n.$$

Hence, we reduced to working with $R = k[x_1, \ldots, x_r]$ and $\mathfrak{n} = (x_1, \ldots, x_r)$.

(1)
- Let $r = 1$. By the above, we have

$$\mathcal{O}\big/\mathfrak{m}^n \cong R\big/\mathfrak{n}^n = k[x]\Big/(x^n)$$

  It is then straightforward to see that on the RHS, $1, x, \ldots, x^{n-1}$ forms a k-basis, and thus we get $\chi(n) = n$.
- Let $r = 2$. Again by the above, we have

$$\mathcal{O}\big/\mathfrak{m}^n \cong R\big/\mathfrak{n}^n = k[x, y]\Big/(x, y)^n.$$

  Again, we see that $\{x^i y^j\}_{i+j<n}$ is a k-basis. Hence we have

$$\chi(n) = |\{(i, j) \in \mathbb{Z}^2_{\geq 0} \mid i + j < n\}| = \sum_{0 \leq i \leq n-1} (i + 1) = \frac{n(n+1)}{2}$$

(2) As above, we have

$$\mathcal{O}/\mathfrak{m}^n \cong R/\mathfrak{n}^n = k[x_1, \ldots, x_r]\Big/(x_1, \ldots, x_r)^n$$

and thus the following is a k-basis:

$$\mathcal{B} := \{x_1^{i_1} \cdots x_r^{i_r} \mid (i_1, \ldots, i_r) \in \mathbb{Z}^r_{\geq 0}, \ \sum_j i_j < n\}$$

Note that by introducing $i_{r+1} = n - 1 - \sum_j i_j$, we obtain

$$|\mathcal{B}| = |\{(i_1, \ldots, i_{r+1}) \in \mathbb{Z}^{r+1}_{\geq 0} \mid \sum_j i_j = n - 1\}|.$$

This agrees with $\dim_k k[x_1, \ldots, x_{r+1}]_{n-1}$, which in Exercise 2 on Sheet 5 we computed to be $\binom{n-1+r}{r}$ (as explained in the solutions for Sheet 5, you can see this by separating $n-1$ stars $*$ with r bars $|$ and constructing a bijection to such arrangements). Therefore, we have

$$\chi(n) = |\mathcal{B}| = \binom{n-1+r}{r},$$

which is a polynomial of degree r with leading coefficient $1/r!$.

Remark. There is also a way to solve the exercise inductively, without computing explicitly $\chi(n)$. Denote by $\chi_r(n)$ the dimension over k of $\mathcal{O}_P(\mathbb{A}^r)/\mathfrak{m}^n$, denote $R_r = k[x_1, \ldots, x_r]$ and $\mathfrak{m}_r = (x_1, \ldots, x_r)$. You can then construct a short exact sequence

$$0 \longrightarrow R_{r+1}/\mathfrak{m}_{r+1}^{n-1} \xrightarrow{\cdot x_{r+1}} R_{r+1}/\mathfrak{m}_{r+1}^{n} \xrightarrow{x_{r+1}=0} R_r/\mathfrak{m}_r^n \longrightarrow 0.$$

Of course, there is some work to do to show that everything is well-defined and exact. Taking this for granted, we then immediately obtain that

$$\chi_{r+1}(n) - \chi_{r+1}(n-1) = \chi_r(n).$$

You can then use this formula to see that if χ_r is a polynomial of degree r with leading coefficient a_r, then χ_{r+1} is a polynomial of degree $r+1$ with leading coefficient $\frac{a_r}{r+1}$.

Exercise 2. Find the multiple points and the tangent lines at the multiple points for each of the following curves:

(1) $X^4 + Y^4 - X^2Y^2$
(2) $X^3 + Y^3 - 3X^2 - 3Y^2 + 3XY + 1$
(3) $Y^2 + (X^2 - 5)(4X^4 - 20X^2 + 25)$

Solution 2. In general, one finds the multiple points by solving the system of equations

$$(*) \qquad \begin{cases} F(x, y) = 0 \\ \frac{\partial F}{\partial X}(x, y) = \frac{\partial F}{\partial Y}(x, y) = 0. \end{cases}$$

(1) $F(X,Y) = X^4 + Y^4 - X^2Y^2$. when solving (*), it becomes quickly clear that we have to distinguish cases according to the characteristic.

<u>char $k = 2$</u> : In this case both partial derivatives are 0, so every point of F is a multiple point. One can also see this by writing

$$F(X,Y) = (X^2 + XY + Y^2)^2.$$

To compute the points on F, as F is homogeneous, we can dehomogenize with $Y = 1$ and find the roots of $X^2 + X + 1$. These are precisely the primitive 3$^{\text{rd}}$ roots of unity, i.e. ζ and ζ^2 (where ζ is an arbitrary choice of primitive 3$^{\text{rd}}$ root of unity). Hence we have

$$F(X,Y) = (X - \zeta Y)^2(X - \zeta^2 Y)^2,$$

i.e. F is the union of two double lines through the origin. So at $(0,0)$, we have two double tangent lines, and at any other point we have one double tangent line.

<u>char $k = 3$</u> : In this case we see that the solutions to (*) are precisely points (x,y) with $x^2 + y^2 = 0$. As in characteristic 3 we have

$$F(X,Y) = (X^2 + Y^2)^2,$$

all of these point lie on F and thus are multiple points. Furthermore, if i denotes a primitive 4$^{\text{th}}$ root of unity, then we see

$$F(X,Y) = (X + iY)^2(X - iY)^2.$$

Hence the picture is similar to the case before: at $(0,0)$ we have two double tangent lines, and at any other point we have one double tangent line.

<u>char $k \neq 2,3$</u> : In this case we see that the only solution to (*) is $(0,0)$, so this is the only multiple point. To find the tangent lines, we need to factor the homogeneous polynomial $F(X,Y)$ into linear terms. To do this, we can dehomogenize to $Y = 1$ and compute the roots of $X^4 - X^2 + 1$. These are precisely $\zeta, -\zeta, \zeta^{-1}, -\zeta^{-1}$ where ζ is a choice of primitive 6$^{\text{th}}$ root of unity (in other words, the roots of $X^4 - X^2 + 1$ are precisely the primitive 3$^{\text{rd}}$ and 6$^{\text{th}}$ roots of unity). Hence we have

$$F(X,Y) = (X - \zeta Y)(X + \zeta Y)(X - \zeta^{-1}Y)(X + \zeta^{-1}Y),$$

i.e. there are four simple tangent lines at $(0,0)$.

(2) $F(X,Y) = X^3 + Y^3 - 3X^2 - 3Y^2 + 3XY + 1$. Again, solving (*) shows that we have to distinguish cases.

<u>char $k = 3$</u> : In this case both partial derivatives vanish, so all points of F are multiple points. We can also see this by writing

$$F(X,Y) = (X + Y + 1)^3.$$

In particular, at every point of F we have the triple tangent line $X + Y + 1$.

char $k \neq 3$: Let us write down (*) in this case

$$x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1 = 0$$
$$3x^2 - 6x + 3y = 0$$
$$3y^2 - 6y + 3x = 0.$$

This lets us write $x^2 = 2x - y$ and $y^2 = 2y - x$. We can then also express x^3 and y^3 with lower powers:

$$x^3 = 2x^2 - xy = 4x - 2y - xy$$
$$y^3 = 2y^2 - xy = 4y - 2x - xy.$$

Plugging all of this into $F(x, y) = 0$, we obtain the equation

$$xy - x - y + 1 = 0,$$

i.e. $(x-1)(y-1) = 0$. So either $x = 1$ or $y = 1$, and then the equations for the partial derivatives force both of them to be equal to 1. One easily checks that $(1, 1)$ is indeed a solution to (*), so this is the unique multiple point of F.

To determine the tangent lines at $(1, 1)$, we compute

$$F(X + 1, Y + 1) = 3XY + X^3 + Y^3.$$

Therefore, $(1, 1)$ is a double point with two distinct tangent lines $X = 0$ and $Y = 0$ (i.e. it is a node).

(3) $F(X, Y) = Y^2 + (X^2 - 5)(4X^4 - 20X^2 + 25)$. Attempting to solve (*) reveals which characteristics we need to treat differently.

char $k = 2$: In this case we have

$$F(X, Y) = (X + Y + 1)^2,$$

so every point of F is a double point with one double tangent line $X + Y + 1$.

char $k = 5$: In this case we have

$$F(X, Y) = Y^2 - X^6.$$

It is then straightforward to see that the only multiple point is at $(0, 0)$, where F has one double tangent line $Y = 0$.

char $k \neq 2, 5$: From $\frac{\partial F}{\partial Y} = 2Y$ we obtain that $y = 0$, and then we need to solve

$$0 = (x^2 - 5)(4x^4 - 20x^4 + 25) = (x^2 - 5)(2x^2 - 5)^2.$$

The solutions to this are precisely $\{\pm\sqrt{5}, \pm\sqrt{5/2}\}$. To see which one of those are multiple points, note that by writing $G(X) = F(X, 0)$, we

have $\frac{\partial F}{\partial X} = \frac{\partial G}{\partial X} = G'$, and so the multiple points of F correspond to the multiple roots of G (recall that for a polynomial G in one variable, x is a multiple root if and only if $G(x) = G'(x) = 0$). From the above factorization, it is clear that these are $\pm\sqrt{5/2}$, i.e. F has the multiple points $(0, \pm\sqrt{5/2})$. In fact, they are double points, as the roots of G are double (and not triple), and because of the Y^2 term.
Finally, to compute the tangent lines at $(0, \pm\sqrt{5/2})$, we have to compute the second homogeneous part of $F(X \pm \sqrt{5/2}, Y)$. To do this efficiently, we take the factorization

$$F(X, Y) = Y^2 - 2(X^2 - 5)(X - \sqrt{5/2})^2(X + \sqrt{5/2})^2$$

and we plug-in $X = \pm\sqrt{5/2}$ fpr every factor which doesn't vanish at $\pm\sqrt{5/2}$. This gives

$$F_2(X \pm \sqrt{5/2}, Y) = Y^2 - 50X^2 = (Y - \sqrt{50}X)(Y + \sqrt{50}X).$$

Hence F has two simple tangent lines at it's multiple points.

Exercise 3. Let $T : \mathbb{A}^2_k \to \mathbb{A}^2_k$ be a polynomial map, $Q \in \mathbb{A}^2_k$ and $P = T(Q)$. If T is written component-wise as (T_1, T_2), the Jacobian matrix of T at Q is defined as $J_Q(T) = (\partial T_i/\partial X_j(Q))_{1 \le i,j \le 2}$.

(1) Show that $m_Q(F^T) \geq m_P(F)$.
(2) Show that if $J_Q(T)$ is invertible, then $m_Q(F^T) = m_P(F)$.
(3) Show that the converse of the previous statement is false.

Solution 3. Let us write T as a column vector, i.e. $T = \binom{T_1}{T_2}$. The key to this exercise is the following observation: for any point $Q = (x, y) \in \mathbb{A}^2$, we can write $T(X + x, Y + y)$ as

$$T(X + x, Y + y) = T(x, y) + J_{(x,y)}(T) \cdot \begin{pmatrix} X \\ Y \end{pmatrix} + \left(X^i Y^j\text{-terms with } i + j \geq 2\right).$$

In other words, both components of $T(X + x, Y + y) - T(x, y) - J_Q(T) \cdot \binom{X}{Y}$ are elements of $(X, Y)^2 \subseteq k[X, Y]$. It suffices to prove this componentwise, i.e. that for all $S \in k[X, Y]$ we have

$$S^{T_Q}(X, Y) = S(X + x, Y + y) - S(x, y) - \left(\frac{\partial S}{\partial X}(x, y)\right) \cdot X - \left(\frac{\partial S}{\partial Y}(x, y)\right) \cdot Y \in (X, Y)^2.$$

This now just follows from the simple observation that $S^{T_Q}(0, 0) = \frac{\partial S^{T_Q}}{\partial X}(0, 0) = \frac{\partial S^{T_Q}}{\partial Y}(0, 0) = 0$, and these are precisely the coefficients of $1, X, Y$ in $S^{T_Q}(X, Y)$.

(1) Let us write $Q = (x_Q, y_Q)$ and $P = T(Q) = (x_P, y_P)$. By the above, we have

$$T(X + x_Q, Y + y_Q) = T(x_Q, y_Q) + S(X, Y) = P + S(X, Y)$$

where if we write $S = \binom{S_1}{S_2}$, then S_1 and S_2 have no constant term. Let us denote by T_P resp. T_Q translation by P resp. Q, i.e $T_P(X,Y) = (X + x_P, Y + y_P)$ and similarly for Q. Then we can rephrase the above by writing

$$T \circ T_Q = T_P \circ S.$$

To compute the multiplicity of F^T at Q, we have to compute the multiplicity of $(F^T)^{T_Q}$, i.e. compute the minimal degree of monomials appearing in it. It is then straightforward to see that

$$(F^T)^{T_Q}(X,Y) = F^{T \circ T_Q}(X,Y) = F^{T_P \circ S}(X,Y) = F^{T_P}(S(X,Y))$$

As the components of $S(X,Y)$ have no constant term, the mnimal degree of monomials in $F^{T_P}(X,Y)$ can only go up if we plug-in $S(X,Y)$. In other words, we conclude that

$$m_Q(F^T) = m_{(0,0)}(F^{T_P \circ S}) \geq m_{(0,0)}(F^{T_P}) = m_P(F).$$

(2) By the general facts explained at the start of the solution, we can write

$$S(X,Y) = \underbrace{J_Q(T) \cdot \begin{pmatrix} X \\ Y \end{pmatrix}}_{L(X,Y):=} + R(X,Y)$$

where the components R_1, R_2 of R are in $(X,Y)^2$. Let $m = m_P(F)$ and consider the degree m part F_m of F^{T_P}. Then we have

$$F_m(S(X,Y)) = F_m(L(X,Y) + R(X,Y)) = F_m(L(X,Y)) + (\text{monomials of degree } > m)\,.$$

So the only way that we could have $m_{(0,0)}(F^{T_p \circ S}) > m_{(0,0)}(F^{T_P})$ is when $F_m(L(X,Y)) = 0$, leaving behind only monomials with higher order. But if $J_Q(T)$ is inveritble, then precomposing with L is an automorphism of $k[X,Y]$ (with inverse given by precomposing with $J_Q(T)^{-1} \cdot \binom{X}{Y}$). Hence in this case we have $F_m(L(X,Y)) \neq 0$, so that

$$m_{(0,0)}(F^{T_P \circ S}) = m = m_P(F).$$

(3) A counterexample is given by $F = Y^2 - X^3$, $P = Q = (0,0)$, $T(X,Y) = (X^2, Y)$. Then $m_Q(F^T) = m_P(F) = 2$ but $J_Q(T) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not invertible.

Exercise 4. Let $n \geq 2$ and $F \in k[X_1, \ldots, X_n]$. Consider $V(F) \subseteq \mathbb{A}_k^n$ the associated hypersurface and $P \in V(F)$.

(1) Define the multiplicity $m_P(F)$ of F at P.
(2) If $m_P(F) = 1$, define the tangent hyperplane of F at P.
(3) Can you define tangent hyperplanes for $F = X^2 + Y^2 - Z^2$ at $P = (0,0,0)$?

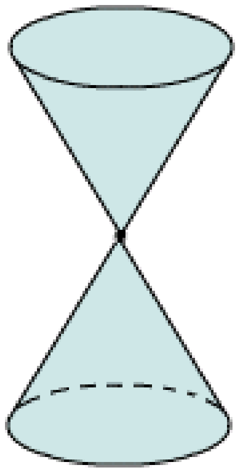
(4) Assume that F is irreducible. Show that, for $n = 2$ (curves), $V(F)$ has finitely many multiple points. Is this true for $n > 2$?

Solution 4.

(1) Let $P = (0, \dots, 0)$. We can define $m_P(F)$ as in the plane curve case. It is the smallest degree m of a summand in the decomposition of F into linear forms. Then for a general P, $m_P(F) = m_0(F^{T_P})$ where T_P is translation by P.

(2) If $m_P(F) = 1$, then $F^T = F_1 + \dots$. The tangent hyperplane of F at P is $V(F_1)$.

(3) In this example, $m_P(F) = 2$, but we cannot factorize $F_2 = F$ into a product of linear forms, as F is irreducible in $k[X, Y, Z]$. So in higher dimensions, the local picture around multiple points is more complicated than simply having unions of lines resp. hyperplanes. Nonetheless, it might be useful to look at $V(F_m)$, i.e. the vanishing locus of the lowest degree form. In fact, this is called the *tangent cone* of F at a point. While 1–dimensional cones are always just unions of lines, higher dimensional cones are more complicated. In our case, the tangent cone $V(F_2)$ of F at $(0, 0)$ is F itself, and you can picture it as a circular double cone through the origin:



(4) Let F be irreducible. The locus of multiple points is given by $V(F, \partial_X F, \partial_Y F)$, where $\partial_X F$ resp. $\partial_Y F$ denote the partial derivatives of F. If $V(F, \partial_X F, \partial_Y F)$ is infinite, it must have dimension ≥ 1, but as $\dim V(F) = 1$, it must have dimension equal to 1. By point (4) of Exercise 1 on Sheet 7, we then obtain $V(F) = V(F, \partial_X F, \partial_Y F)$. In particular, we have

$$\partial_X F, \partial_Y F \in I(V(F)) \underset{\substack{\uparrow \\ \boxed{F \text{ irreducible}}}}{=} (F)$$

But $\partial_X F, \partial_Y F$ are of smaller degree than F, and thus we must have $\partial_X F = \partial_Y F = 0$. In characteristic 0, this gives that F is constant, and thus not a curve (by definition). On the other hand, if $\operatorname{char} k = p$, then the partial derivatives can only vanish simultaneously if F is of the form

$$F(X,Y) = G(X^p, Y^p)$$

for some $G \in k[X,Y]$. But then if we write $G = \sum_{i,j} G_{i,j} X^i Y^j$ and $G^{1/p} := \sum_{i,j} G_{i,j}^{1/p} X^i Y^j$ (as k is algebraically closed, we can take p-th roots), we obtain

$$F(X,Y) = (G^{1/p}(X,Y))^p,$$

so F is not irreducible, contradiction.

For $n > 2$, we have a counterexample given by taking the product of any irreducible singular curve with $\mathbb{A}^1$. That is, if $F \in k[X,Y]$ is an irreducible curve with a multiple point $P = (x_P, y_P)$, then considering F as an element of $F \in k[X,Y,Z]$, we obtain an irreducible hypersurface where every point of the form (x_P, y_P, z) is multiple. However, the same proof as above shows that the locus of multiple points $V(F, \partial_{X_i} F \mid 1 \leq i \leq n)$ is always a strict closed subset of $V(F)$, i.e. it has codimension at least 1.

Exercise 5. Let $R = k[\epsilon]/(\epsilon^2)$ and $\varphi : R \to k$ the k-algebra homomorphism sending ϵ to 0 (R is often called the ring of *dual numbers*). Let $F \in k[X,Y]$ irreducible, $P \in V(F)$, $\mathfrak{m}_P \subseteq \Gamma(F)$ the corresponding maximal ideal and $\theta_P : \Gamma(F) \to \Gamma(F)/\mathfrak{m}_P \simeq k$ the associated k-algebra homomorphism.

(1) Suppose that P is a simple point. Show that there is a bijection between the tangent line to F at P and $\{\theta \in Hom_{k-alg}(\Gamma(F), R) \mid \varphi \circ \theta = \theta_P\}$.
(2) What happens for multiple points (for instance, $F = Y^2 - X^3$, $P = (0,0)$)?

Solution 5.

(1) Denote $P = (x_P, y_P)$ and by T_P translation by P. Note that as F is irreducible, we have $\Gamma(F) = k[X,Y]/(F)$. Denote by $\overline{X}$ and $\overline{Y}$ the classes of X resp. Y in $\Gamma(F)$. Also, by abuse of notation, denote the class of ϵ inside R by ϵ as well.

From the isomorphism theorems, k-algebra homomorphisms from $\Gamma(F)$ to R correspond to k-algebra homomorphisms from $k[X,Y]$ to R sending F to 0.

Let $\theta \colon \Gamma(F) \to R$ be a k-algebra homomorphism such that $\varphi \circ \theta = \theta_P$. We then obtain

$$x_P = \theta_P(\overline{X}) = \varphi \circ \theta(\overline{X})$$

and similarly $y_P = \varphi \circ \theta(\overline{Y})$. Hence, θ is of the form

$$\theta \colon \Gamma(F) \to R$$
$$\overline{X} \mapsto x_P + a\epsilon, \ \overline{Y} \mapsto y_P + b\epsilon$$

for some $a, b \in k$. Now notice the following: we have

$$0 = \theta(F(\overline{X}, \overline{Y}) = F(\theta(\overline{X}), \theta(\overline{Y})) = F(x_P + a\epsilon, x_Q + b\epsilon) = F^{T_P}(a\epsilon, b\epsilon).$$

Now by writing $F = F_1 + F_2 + \cdots$, we obtain

$$\begin{aligned} 0 &= F_1(a\epsilon, b\epsilon) + F_2(a\epsilon, b\epsilon) + \cdots \\ &= F_1(a,b)\epsilon + \underbrace{F_2(a,b)\epsilon^2 + \cdots}_{=0} \\ &= F_1(a,b)\epsilon. \end{aligned}$$

Therefore, it follows that $(a, b) \in V(F_1)$, i.e. (a, b) is a point on the tangent line of F at P.

In conclusion, if we denote

$$T_P^{\mathrm{Zar}}(F) \coloneqq \{\theta \in \mathrm{Hom}_{k-\mathrm{alg}}(\Gamma(F), R) \mid \varphi \circ \theta = \theta_P\}$$

and

$$\begin{aligned} \mathrm{pr}_\epsilon \colon R &\to k \\ a_0 + a_1\epsilon &\mapsto a_1, \end{aligned}$$

we obtain a map

$$\begin{aligned} \Phi \colon T_P^{\mathrm{Zar}}(F) &\to V(F_1) \\ \theta &\mapsto (\mathrm{pr}_\epsilon(\theta(\overline{X})), \mathrm{pr}_\epsilon(\theta(\overline{Y}))). \end{aligned}$$

To obtain a map in the reverse direction, fix $(a, b) \in V(F_1)$. We can then define a morphism of k–algebras by

$$\begin{aligned} \Theta \colon k[X,Y] &\to R \\ X \mapsto x_P + a\epsilon&, \; Y \mapsto y_P + b\epsilon. \end{aligned}$$

By the same computation as above, we then have $\Theta(F) = F(x_P + a\epsilon, y_P + b\epsilon) = 0$, and thus we obtain a morphism of k–algebras

$$\begin{aligned} \theta_{a,b} \colon k[X,Y] &\to R \\ \overline{X} \mapsto x_P + a\epsilon&, \; \overline{Y} \mapsto y_P + b\epsilon. \end{aligned}$$

It is then clear that $\theta \in T_P^{\mathrm{Zar}}(F)$. Hence we obtain a map

$$\begin{aligned} \Psi \colon V(F_1) &\to T_P^{\mathrm{Zar}}(F) \\ (a,b) &\mapsto \theta_{a,b}, \end{aligned}$$

and it is straightforward to check that the two constructions are muttually inverse.

(2) For the cusp, for all $a, b \in k^2$, $F(a\epsilon, b\epsilon) = 0$ because $\epsilon^2 = 0$. Hence, we can perform the above construction to obtain a bijection between $T_{(0,0)}^{\mathrm{Zar}}(F)$ and k^2.

Remark. The space $T_P^{\mathrm{Zar}}(F)$ is called the *Zariski tangent space* of F at P. One can in fact endow it with a natural vector space structure, compatible with the above bijections. It is in fact always the case that for $F \in k[X,Y]$, we have $\dim_k T_P^{\mathrm{Zar}}(F) = 1$ if and only if P is a simple point. Note that one can define $T_P^{\mathrm{Zar}}(V)$ for an affine algebraic variety $V \subseteq \mathbb{A}^n$ in the same way. In this case, it corresponds to the affine subspace of $\mathbb{A}^n$ generated by all the tangent directions of V at P.
The definition of $T_P^{\mathrm{Zar}}(F)$ has actually a very concrete motivation coming from differential geometry, let me try to explain it: the most important fact about affine algebraic sets is that they are, up to isomorphism, in a one-to-one correspondence with finitely generated, reduced k-algebras, and this correspondence is compatible with morphisms:

$$\{\text{affine algebraic sets}\}\big/\cong \;\overset{1:1}{\longleftrightarrow}\; \{\text{finitely generated, reduced } k\text{-algebras}\}\big/\cong$$

see Proposition 2.5 in the lecture notes (for those familiar with the language, the precise mathematical term for this is an *equivalence of categories*). In words, an affine algebraic set V corresponds to its coordinate ring $\Gamma(V)$, i.e. loosely speaking the space of functions $V \to k$. Now what happens if we drop the word 'reduced' from the right hand side above? Could we still regard some non-reduced, finitely generated k-algebra as the space of functions from some geometric object to k? In fact, we can!
For the non-reduced k-algebra $R = k[\epsilon]/(\epsilon^2)$, you should think of it as the space of functions from an infinitesimally small neighborhood around 0 (very loosely speaking, in $\mathbb{R}$, think of the open interval $(-\epsilon, \epsilon)$ as $\epsilon \to 0^+$). Remember how in Analysis 1 and 2, there were always these proofs where terms with an ϵ^2 can be thrown away when $\epsilon \to 0$? Well here it's the same idea, only that for us, we in fact have $\epsilon^2 = 0$ in R. Let me denote by I_ϵ this infinitesimally small interval around 0 (without really saying what it is), such that R is the space of functions from I_ϵ to k. Then what does

$$T_P^{\mathrm{Zar}}(F) = \{\theta \in \mathrm{Hom}_{k-\mathrm{alg}}(\Gamma(F), R) \mid \varphi \circ \theta = \theta_P\}$$

correspond to geometrically? Well, if we take for granted that there still is a correspondence as above for non-reduced things, then $\mathrm{Hom}_{k-\mathrm{alg}}(\Gamma(F), R)$ should correspond to $\mathrm{Hom}(I_\epsilon, F)$, i.e. maps from I_ϵ into the curve F. The condition $\varphi \circ \theta = \theta_P$ then translates into the condition that $0 \in I_\epsilon$ is mapped to P. So geometrically, we can regard $T_P^{\mathrm{Zar}}(F)$ as the set of funtions from a small neighborhood around 0 to F, which map 0 to P. Does that ring a bell? It is supposed to model a construction from differential geometry, where if you have a point P in a smooth manifold M, each tangent vector $v \in T_P M$ can be obtained from a smooth curve $\gamma\colon (-\epsilon, \epsilon) \to M$ sending 0 to P, such that $v = \gamma'(0)$.